

# Neutron Skin size dependence of the nuclear binding energy

S.J. Lee<sup>1,2</sup> and A.Z. Mekjian<sup>2</sup>

<sup>1</sup>*Department of Applied Physics, Kyung Hee University, Yongin-si, Gyeonggi-do 446-701, Korea and*

<sup>2</sup>*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854, USA*

The nuclear binding energy is studied using a finite temperature density functional theory. A Skyrme interaction is used in this work. Volume, surface, and symmetry energy contributions to the binding energy are investigated. The case of neutron skin is considered in detail. The neutron skin modifies the mass  $A$  dependence of various terms and  $I$  dependence of the skin thickness is proportional to  $I$  for the case of same central density.

## I. INTRODUCTION

The mass formulae [1–4] characterizes the binding energy of nuclei in terms of the proton number  $Z$  and neutron number  $N$  or nucleon number  $A = Z + N$ . Volume, surface, Coulomb and pairing terms appear. Of importance and one of the least determined parts of the binding energy expression is the symmetry energy term. The symmetry energy has both a volume symmetry part and a surface symmetry part, similar to the division of the volume and surface terms. The division into volume and surface energies occurs because observed nuclei have finite nucleon number with a maximum value of  $A \sim 250$ . The symmetry energy appears in the total binding energy with factors involving both  $((N - Z)/A)^2 A$  and  $((N - Z)/A)^2 A^{2/3}$  for the volume and surface symmetry terms respectively. For a heavy nucleus with  $A = 200$ ,  $Z = 80$ ,  $N = 120$  the  $(N - Z)/A = 1/5$  and  $((N - Z)/A)^2 = 1/25$ . The isospin dependence of the binding energy arises in part from kinetic energy differences between protons and neutrons and from interaction terms arising from the isospin dependence of the nuclear force between nucleons. This interaction has terms involving the  $\vec{\tau}_i \cdot \vec{\tau}_j$  which arises from the exchange of isovector mesons. The surface terms in the mass formulae arise from the loss of binding energy for nucleons near the surface of a nucleus. In a neutron star, the volume term only exists. Thus, of importance for neutron star physics is the extraction of the volume term from properties of known finite nuclei which contain both volume and surface terms. The symmetry term also plays a significant role in heavy ion collisions [5–7], in neutron star physics [8], in the valley of nuclear stability and associated neutron and proton drip lines, in neutron halo nuclei in isobaric analog states [9–11], and in giant isovector dipole states [2]. Extensive discussion of the symmetry energy can be found in the work of Danielewicz et al. [12]. The role of the symmetry energy in the nuclear surface has also been studied in Ref.[12]. An extensive review can be found in Ref.[8].

In our previous paper [13], we have examined the  $T$  dependence of the expansion coefficients in mass formulae by minimizing the Helmholtz free energy using Skyrme interaction. In there we have assumed  $R_n = R_p$  and thus missed any effects of neutron skin which would exist in asymmetric nuclei. We estimated approximately neutron skin effect for  $T^2$  term in kinetic energy by setting the central densities to be the same  $\rho_{nc} = \rho_{pc}$  and found the effect of  $R_n \neq R_p$  is small. But this is only a part of neutron skin effect thus we will study the neutron skin dependence more fully here by expanding energy with the neutron skin size  $R_n - R_p$ .

In this paper we study the neutron skin effect on the volume and surface contributions to the symmetry energy. Our approach is based on a finite temperature density functional theory and we use a Skyrme type of interaction which we develop in the next section. The same Skyrme approach was also used to study phase transition in Ref.[14–17]. The symmetry energy and the neutron skin effects on symmetry energy and other energy coefficients for three cases of Skyrme interactions are studied in Sect. III and the results are summarized in Sect. IV.

## II. BINDING ENERGY IN A DENSITY FUNCTIONAL APPROACH.

A density functional theory based on a Skyrme interaction will be used in our investigation which is summarized in Appendix A. The density distribution is taken to be of the Saxon-Wood form:

$$\rho_q(\vec{r}) = \rho_q(R_q) = \frac{\rho_{qc}}{1 + e^{(r-R_q)/a_q}} \quad (1)$$

The  $q = p, n$  for protons, neutrons. We will allow the central density  $\rho_{qc}$  and the  $R_q$  for protons and neutrons to be different in general. The diffuseness  $a_q$  are taken to be the same. Two limiting cases can also be considered. These are: 1.  $R_p = R_n$  so that the difference in  $N \neq Z$  nuclei is in the central density or  $\rho_{pc} \neq \rho_{nc}$  and 2.  $R_p \neq R_n$  and  $\rho_{pc} = \rho_{nc}$ . In case 2 the neutron distribution reflects a neutron halo for  $N > Z$ .

Using this density functional approach the binding energy of nuclei as a function of mass number  $A$ , proton number  $Z$  and temperature  $T$  can be evaluated. The Weizacker semiempirical mass formulae [1], its extension by Myers and

Swiatecki [18–21] and also including finite temperature effects [13] is

$$E(A, Z, T) = E_V(T)A + E_S(T)A^{2/3} + S_V(T)I^2A + S_S(T)I^2A^{2/3} \\ + E_C \frac{Z^2}{A^{1/3}} + E_{dif} \frac{Z^2}{A} + E_{ex} \frac{Z^{4/3}}{A^{1/3}} + c\Delta A^{-1/2} \quad (2)$$

where  $I = (N - Z)/A = (A - 2Z)/A$ . The  $B = -E_V$  is the usual bulk energy per nucleon. The  $E_{dif}$  and  $E_{ex}$  are the coefficients for the diffuseness correction and the exchange correction to the Coulomb energy. For the pairing correction with constant  $\Delta$ ,  $c = +1$  for odd-odd nuclei, 0 for odd-even nuclei, and  $-1$  for even-even nuclei. The above formula at  $T = 0$  is the well known Weizacker semiempirical mass formula [1–4] studied extensively by Myers and Swiatecki [18–21]. Early studies excluded the surface symmetry term  $S_S$  and only the surface term  $E_S$  was included. The values of the coefficients as found in textbooks such as Ref.[2, 4] are  $E_V(0) = -B(0) \approx -16$ ,  $E_S(0) \approx 17$ , and  $S_V(0) \approx 24$  in MeV. The ratio  $E_S/B$  of surface to bulk energy at  $T = 0$  is very close to unity.

In our preveous paper [13], we have examined the  $T$  dependence of the expansion coefficients in Eq.(2) by minimizing the Helmholtz free energy using a Skyrme interaction with the density distribution of Eq.(1). With  $R_n = R_p = R$ , we can integrate Skyrme interaction analytically to obtain total energy as a function of  $R$ . Using this energy function, the total energy  $E(A, Z, T)$  minimizing free energy is found by varying the value of  $R$  for a nucleus with  $Z$  protons and  $N$  neutrons at temperature  $T$ . Then we use Eq.(2) for various nuclei to obtain the expansion coefficients. Since we have assumed  $R_n = R_p$  in our previous studies, we missed any effects of neutron skin which would exist in asymmetric nuclei. We estimated approximately neutron skin effect for  $T^2$  term in kinetic energy, Eq.(A3), by setting the central densities to be the same,  $\rho_{nc} = \rho_{pc} = \rho_c/2$  where  $\rho_c$  is the total central density obtained with  $R_n = R_p$ , and found the effect of  $R_n \neq R_p$  in kinetic energy is small.

In the present paper we now consider neutron skin effects more fully. Specifically, we study the neutron skin size  $t = R_n - R_p$  dependence of total energy  $E(A, Z, T, t)$ . Then we expand each coefficient  $E_i$  of Eq.(2) as

$$E_i(T, t) = E_i(T) + E_i(T)_{sk} \frac{t}{A^{1/3}} \quad (3)$$

Since we can integrate a Skyrme interaction analytically for the case of  $R_n = R_p = R$ , we can obtain the neutron skin  $t$  dependence by expanding the integral for the case of  $R_n \neq R_p$  around  $R_n = R_p = R$ . This can be done by expanding the density distribution of Eq.(1) around  $R$  which we now discuss. It should also be noted that in Eq.(3) the surface correction appears with the factor  $t/A^{1/3}$  which originates from the dimensionless factor  $t/R$  with  $R = r_0 A^{1/3}$ . The extra  $1/A^{1/3}$  introduces a different  $A$  dependence from the non skin part  $E_i(T)$ .

From Eqs.(2) and (3) we see that the surface thickness factor  $t/A^{1/3}$  changes the  $A$  dependences in the mass formular. In particular  $AE_i(T)_{sk}t/A^{1/3}$  acts as a surface term as  $A^{2/3}$  and  $A^{2/3}E_i(T)_{sk}t/A^{1/3}$  goes like  $A^{1/3}$  when we look at the whole  $A$  dependences of the neutron skin size dependent coefficients. When  $R_q \approx R$ , the neutron skin size  $t/R$  is proportional to  $\frac{A^2}{NZ}$  as can be seen in Appendix B. Thus the skin dependence of the energy expansion coefficients have this extra factor of  $A$  dependence. On the other hand, when neutron and proton central densities are the same, the neutron skin size  $t/R$  is proportional to  $(N - Z)/A$  (see Appendix B). Then the skin dependence of the energy expansion coefficients introduce an extra  $I = (N - Z)/A$  behavior and the energy expansion of Eq.(2) with Eq.(3) becomes a third power expansion in the isospin factor  $I$ . That is  $t/A^{1/3} \sim I$  since  $t/R \sim I$ . For such a case, by expanding the empirical nuclear energy for various nuclei together with  $I$  and  $I^3$  terms included, we may be able to extract the information about the neutron skin size from the odd term in  $I$ .

The central density  $\rho_{qc}$  of the density distribution Eq.(1) for a given value of  $R_q$  should be determined to give a fixed number of nucleons  $N_q$ . This normaization condition gives the expansion of  $\rho_{qc}(R_q)$  as given by Eq.(B4) in Appendix B up to first order in  $t_q = (R_q - R)$ . Thus the first order correction coefficient to  $\rho_{qc}^m$  due to fixed  $N_q$  is the zeroth order term times  $-\frac{m}{R} \left[ \frac{3+\pi^2(\frac{A}{R})^2}{1+\pi^2(\frac{A}{R})^2} \right]$ . This result is independent of which type of particle, neutron or proton. Furthermore if we keep the particle number  $N_q$  and the total central density  $\rho_c = \rho_{nc} + \rho_{pc}$  to be a constant while varying  $R_q$ , that is,

$$\rho_c(R) = \rho_{nc}(R_n) + \rho_{pc}(R_p) = \rho_{nc}(R) + \rho_{pc}(R) \quad (4)$$

then we have

$$t_n = R_n - R = \frac{Z}{A}t, \\ t_p = R_p - R = -\frac{N}{A}t \quad (5)$$

to lowest order in  $t = R_n - R_p$  when we expand about  $R_q = R$  (see Appendix B).

The Fermi density  $\rho_q(r)$  is then expanded about  $R_q = R$  up to first order in  $(R_q - R)$  as given by Eqs.(B18)-(B20). The results of Appendix B show that the quantity  $\rho_n^m(r) + \rho_p^m(r)$  has a first order correction from skin size  $t$  and the first order correction vanishes for  $m = 1$ . That is the total density  $\rho(r) = \rho_n(r) + \rho_p(r)$  is independent of the neutron skin size  $t = R_n - R_p$  up to first order. Thus only the explicit  $\rho_q$  dependent terms in Skyrme interaction, not the total density  $\rho$  dependent terms, depend on the skin size  $t$  up to the first order.

Since  $F[\rho_q] = \int d^3r f(\rho_q(\vec{r}))$  for a Fermi density  $\rho_q(r) = \frac{\rho_{qc}}{1 + e^{(r-R_q)/a}}$ , where  $f(\rho_q)$  is only a function of single density  $\rho_q$ , then  $F[\rho_q]$  can be integrate exactly. The result is a function  $F(R_q)$  of  $R_q$  [13], which we can expand easily in terms of  $t_q = R_q - R$  around  $F(R)$ . That is

$$F(R_q) = F(R) + \frac{dF(R)}{dR} t_q + \frac{d^2F(R)}{dR^2} \frac{t_q^2}{2} + \dots \quad (6)$$

This procedure is much simpler than using a method of expanding the density in  $t$  first and then integrating the results. For fixed  $N_q$ , the central density  $\rho_{qc}(R_q)$  is also a function of  $R_q$  as in Eq.(B4). The expansion of integral of various power of  $\rho_q$  are summarized in Appendix C.

For the integral of the form of  $\int d^3r \rho_q^m$ , we need to expand  $\rho_q^m(R_q)$  around  $R$  first then integrate each term which is now a function of  $R$  only. The results for various cases are also given in Appendix C.

### III. NEUTRON SKIN SIZE DEPENDENCE

Here we examine the neutron skin size dependence of nuclear energy using various Skyrme interactions. We used three sets of Skyrme parameters with different values of the effective mass in symmetric nuclear matter which are SLy4 with  $m^*/m = 0.69$ , SkM\* with  $m^*/m = 0.79$ , and SkM( $m^* = m$ ) with  $m^*/m = 1$ . The results for these three cases are given in Table I and summarized in the following equations. The three cases cover a wide range of interaction types in terms of effective mass within the many various Skyrme interactions.

For Coulomb energy, from Eqs.(C8) and (C5) with  $a = 0.53$  fm and  $R = 1.25A^{1/3}$  fm,  $E_C$  part is 0.6912000,  $E_{ex}$  part is -0.5278064 and  $E_{diff}$  -1.430810 and the skin dependence of Coulomb energy  $E_C$  part is -0.5529600,  $E_{ex}$  part is 0.4222251 and  $E_{diff}$  3.433944, that is

$$\begin{aligned} E_C(T, t) \frac{Z^2}{A^{1/3}} + E_{diff}(T, t) \frac{Z^2}{A} + E_{ex}(T, t) \frac{Z^{4/3}}{A^{1/3}} = & \left[ 0.6912 \frac{Z^2}{A^{1/3}} - 1.4308 \frac{Z^2}{A} - 0.5278 \frac{Z^{4/3}}{A^{1/3}} \right] \\ & + \left[ -0.5530 \frac{Z^2}{A^{1/3}} + 3.4339 \frac{Z^2}{A} + 0.4222 \frac{Z^{4/3}}{A^{1/3}} \right] \frac{t}{A^{1/3}} \end{aligned} \quad (7)$$

in MeV and fm units.

In Table I, the items labeled “ $E_i(T)$ ” and “ $E_i(T)_{sk}$ ” are the energy expansion coefficients given in Eqs.(2) and (3). The items labeled by “ $T$ -indp”, “ $T^2$ ”, and “Kine” under  $E_i(T)$  are the values of temperature  $T$  independent part,  $T^2$  dependent term in kinetic energy (Eq.(A3)), and the total kinetic energy contribution to the energy expansion coefficients  $E_i(T)$  respectively. Similarly the items labeled by “ $T$ -indp”, “ $T^2$ ”, and “Kine” under  $E_i(T)_{sk}$  are the values of temperature  $T$  independent part,  $T^2$  dependent term in kinetic energy (Eq.(A3)), and the total kinetic energy contribution to the neutron skin dependent part of the energy expansion coefficients  $E_i(T)_{sk}$  respectively.

The results of Table I show the following features for the skin size dependence of various coefficients  $E_i(T)_{sk}$  of Eq.(3) in the mass formulae which behave as  $E_i(T)_{sk} t/A^{1/3}$ . One overall feature for all components  $E_i(T)_{sk}$  is the weak dependence on temperature. The volume binding energy coefficient  $E_i(T)_{sk} = E_V(T)_{sk}$  of about 5 ~ 7 MeV/fm has the smallest value of all the  $E_i(T)_{sk}$  terms. Comparing to this value, the skin independent volume energy coefficient  $E_V(T)$  is about -15 MeV. The surface energy coefficient  $E_S(T)_{sk}$  has a somewhat larger values of about -17 ~ -22 MeV/fm and is negative compared to about 20 MeV for skin independent coefficient  $E_S(T)$ . The neutron skin size dependence of the symmetry energy terms have the following features. The volume symmetry energy coefficient  $S_V(T)_{sk} \sim 60$  MeV/fm while the surface symmetry energy coefficient has the largest magnitude of about -600 ~ -700 MeV/fm. Comparing to these the neutron skin size independent coefficients are  $S_V(T) \approx 20 \sim 30$  MeV and  $S_S(T) \approx 30 \sim 50$  MeV. Some dependence on the effective mass is present for all coefficients as can be seen in comparing SLy4 ( $m^*/m = 0.7$ ), SkM\* ( $m^*/m = 0.8$ ), and SkM( $m^* = m$ ). However the dependences of the individual coefficients on the temperatures and on the Skyrme parameters we used are not so sensitive. The magnitude of the neutron skin size dependent coefficients are largely different between different coefficients ranging from about 5 MeV/fm for volume energy coefficient to about 700 MeV/fm for surface symmetry energy coefficient while the magnitudes of the skin size independent parts were of the same order of magnitude ranging from about 15 MeV for

TABLE I: Neutron Skin Dependence of Energy coefficient minimizing free energy.

	SLy4				SkM*				SkM( $m^* = m$ )			
$T$ (MeV)	0	1	2	3	0	1	2	3	0	1	2	3
$E_V(T)$	-15.308	-15.296	-15.263	-15.217	-15.127	-15.108	-15.051	-14.962	-15.310	-15.270	-15.152	-14.954
$T$ -indp	-15.308	-15.308	-15.308	-15.312	-15.127	-15.127	-15.127	-15.130	-15.310	-15.310	-15.310	-15.310
$T^2$	0.01151	0.01145	0.01121	0.01061	0.01925	0.01921	0.01908	0.01868	0.03948	0.03948	0.03950	0.03948
Kine	28.653	28.594	28.423	28.124	25.483	25.444	25.319	25.105	19.957	19.978	20.037	20.134
$E_V(T)_{sk}$	5.5202	5.5100	5.4792	5.42745	5.5837	5.5730	5.5411	5.48746	6.9809	6.9701	6.9379	6.88390
$T$ -indp	5.5202	5.5172	5.5081	5.4927	5.5837	5.5806	5.5717	5.5563	6.9809	6.9780	6.9697	6.9553
$T^2$	-0.00722	-0.00722	-0.00723	-0.00725	-0.00761	-0.00762	-0.00763	-0.00765	-0.00793	-0.00793	-0.00793	-0.00794
Kine	-3.8780	-3.8654	-3.8269	-3.7576	-0.4893	-0.4995	-0.5302	-0.5816	-0.3867	-0.3963	-0.4253	-0.4737
$E_S(T)$	20.008	20.559	22.247	25.204	18.756	19.296	20.948	23.836	18.303	18.804	20.321	22.904
$T$ -indp	20.008	20.009	20.024	20.099	18.756	18.757	18.771	18.841	18.303	18.304	18.310	18.339
$T^2$	0.5483	0.5501	0.5559	0.5672	0.5371	0.5388	0.5443	0.5550	0.4995	0.5003	0.5027	0.5072
Kine	-32.106	-31.638	-30.255	-27.943	-26.275	-25.810	-24.402	-22.060	-15.845	-15.379	-13.966	-11.598
$E_S(T)_{sk}$	-17.077	-17.040	-16.928	-16.739	-17.261	-17.222	-17.103	-16.903	-21.691	-21.656	-21.550	-21.372
$T$ -indp	-17.077	-17.059	-17.006	-16.915	-17.261	-17.243	-17.187	-17.090	-21.691	-21.678	-21.638	-21.569
$T^2$	0.01959	0.01958	0.01957	0.01955	0.02082	0.02082	0.02082	0.02080	0.02201	0.02201	0.02200	0.02199
Kine	12.325	12.290	12.179	11.981	1.4093	1.4377	1.5238	1.6681	1.1228	1.1493	1.2296	1.3638
$S_V(T)$	31.113	31.664	33.401	36.645	29.655	30.210	31.960	35.221	19.685	20.110	21.412	23.695
$T$ -indp	31.113	31.115	31.150	31.333	29.655	29.657	29.693	29.875	19.685	19.686	19.698	19.756
$T^2$	0.5445	0.5489	0.5626	0.5902	0.5482	0.5527	0.5667	0.5940	0.4227	0.4241	0.4287	0.4377
Kine	-24.674	-24.568	-24.103	-23.455	-33.636	-33.481	-32.837	-31.703	-6.8813	-6.5197	-5.4723	-3.7475
$S_V(T)_{sk}$	63.203	63.225	63.266	63.306	62.787	62.836	62.956	63.125	63.366	63.381	63.428	63.498
$T$ -indp	63.203	63.194	63.140	63.019	62.787	62.806	62.836	62.852	63.366	63.350	63.301	63.209
$T^2$	0.03095	0.03104	0.03134	0.03188	0.02954	0.02962	0.02988	0.03037	0.03144	0.03151	0.03171	0.03207
Kine	1.3320	1.3578	1.3656	1.3511	-1.3400	-1.2880	-1.1223	-0.8333	-0.8709	-0.8224	-0.6789	-0.4350
$S_S(T)$	-41.035	-43.642	-51.815	-66.958	-43.596	-46.185	-54.328	-69.369	-33.175	-35.246	-41.591	-52.721
$T$ -indp	-41.035	-41.046	-41.196	-41.984	-43.596	-43.606	-43.767	-44.551	-33.175	-33.179	-33.234	-33.522
$T^2$	-2.5769	-2.5962	-2.6548	-2.7749	-2.5598	-2.5793	-2.6404	-2.7575	-2.0606	-2.0668	-2.0893	-2.1332
Kine	171.020	170.079	165.833	159.697	166.818	165.864	162.452	155.996	56.5082	54.7216	49.7064	41.5131
$S_S(T)_{sk}$	-711.559	-711.507	-711.128	-710.053	-706.246	-706.252	-706.059	-705.169	-611.848	-612.332	-613.788	-616.101
$T$ -indp	-711.559	-710.590	-707.436	-701.662	-706.246	-705.310	-702.271	-696.560	-611.848	-611.377	-609.957	-607.442
$T^2$	-0.91606	-0.91767	-0.92289	-0.93232	-0.94030	-0.94192	-0.94708	-0.95653	-0.95456	-0.95538	-0.95785	-0.96219
Kine	-276.281	-275.271	-271.698	-265.302	-40.3526	-41.5910	-45.3514	-51.7077	-31.5405	-32.7350	-36.3085	-42.3087

volume energy to about 50 MeV for surface symmetry energy coefficient. The neutron skin size dependence of surface symmetry energy is much larger than the skin dependence of volume symmetry energy with  $A$  dependence included even for a large  $A$  of over 200. We can also see that the neutron skin size dependent coefficients for volume energy and surface energy  $E_V(T)_{sk}$  and  $E_S(T)_{sk}$  have opposite sign compare to the corresponding skin size independent coefficients  $E_V(T)$  and  $E_S(T)$ , while volume symmetry energy and surface symmetry energy  $S_V(T)$  and  $S_S(T)$  have the same sign for neutron skin size dependent coefficients and independent coefficients.

The kinetic energy contribution to the neutron skin size dependent and independent coefficients have a much more sensitive dependence on the Skyrme parameter set we used. The magnitude of kinetic energy contribution follows somewhat the magnitude of the effective mass of the Skyrme parameter set used. Since the neutron skin size dependent coefficients  $E_i(T)_{sk}$  themselves are somewhat insensitive to the Skyrme parameter set used, the potential energy contribution to the coefficients, which are function of density, also are sensitive to the parameter set used. The kinetic energy contribution to the neutron skin size dependence  $E_i(T)_{sk}$  are now opposite in sign to the kinetic energy part to the skin size independent coefficients  $E_i(T)$  of energy expansion for all the terms. In turn, they are all opposite sign to the skin size independent coefficients  $E_i(T)$  themselves. The kinetic energy contribution to the neutron skin size dependent volume symmetry energy coefficient  $S_V(T)_{sk}$  has a small magnitude similar to the magnitude of the kinetic energy contribution to the skin size dependent volume energy coefficient  $E_V(T)_{sk}$ . The

kinetic energy contribution to the surface symmetry energy coefficients, both the neutron skin size dependent and independent ones  $S_S(T)_{sk}$  and  $S_S(T)$ , have largest magnitude among the energy expansion coefficients. It is much larger even with  $A$  dependence included than skin size dependence of volume symmetry energy  $S_V(T)_{sk}$  and other energy expansion coefficients.

When the Helmholtz free energy is minimized, from the values of “ $T$ -indp” and “ $T^2$ ” for  $T = 0$  in Table I, the temperature and neutron skin size dependence of the energy at low  $T$  becomes

$$\begin{aligned}
E(A, Z, T, t) = & \left[ -(15.308 - 0.012T^2)A + (20.008 + 0.548T^2)A^{2/3} \right. \\
& \left. + (31.113 + 0.545T^2)I^2A - (41.035 + 2.577T^2)I^2A^{2/3} \right] \\
& + \left[ (5.520 - 0.007T^2)A - (17.077 - 0.020T^2)A^{2/3} \right. \\
& \left. + (63.203 + 0.031T^2)I^2A - (711.559 + 0.916T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}} \\
& + E_C(T, t) \frac{Z^2}{A^{1/3}} + E_{dif}(T, t) \frac{Z^2}{A} + E_{ex}(T, t) \frac{Z^{4/3}}{A^{1/3}}
\end{aligned} \tag{8}$$

for SLy4 parameter set,

$$\begin{aligned}
E(A, Z, T, t) = & \left[ -(15.127 - 0.019T^2)A + (18.756 + 0.537T^2)A^{2/3} \right. \\
& \left. + (29.655 + 0.548T^2)I^2A - (43.596 + 2.560T^2)I^2A^{2/3} \right] \\
& + \left[ (5.584 - 0.008T^2)A - (17.261 - 0.021T^2)A^{2/3} \right. \\
& \left. + (62.787 + 0.030T^2)I^2A - (706.246 + 0.940T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}} \\
& + E_C(T, t) \frac{Z^2}{A^{1/3}} + E_{dif}(T, t) \frac{Z^2}{A} + E_{ex}(T, t) \frac{Z^{4/3}}{A^{1/3}}
\end{aligned} \tag{9}$$

for SkM\* parameter set, and

$$\begin{aligned}
E(A, Z, T, t) = & \left[ -(15.310 - 0.039T^2)A + (18.303 + 0.500T^2)A^{2/3} \right. \\
& \left. + (19.685 + 0.423T^2)I^2A - (33.175 + 2.061T^2)I^2A^{2/3} \right] \\
& + \left[ (6.981 - 0.008T^2)A - (21.691 - 0.022T^2)A^{2/3} \right. \\
& \left. + (63.366 + 0.031T^2)I^2A - (611.848 + 0.955T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}} \\
& + E_C(T, t) \frac{Z^2}{A^{1/3}} + E_{dif}(T, t) \frac{Z^2}{A} + E_{ex}(T, t) \frac{Z^{4/3}}{A^{1/3}}
\end{aligned} \tag{10}$$

for SkM( $m^* = m$ ) parameter set in MeV and fm units.

From the values of “ $T$ -indp” and “ $T^2$ ” for  $T = 0$  in Table I, the temperature and neutron skin size dependence of the kinetic energy at low  $T$  becomes

$$\begin{aligned}
E_K(A, Z, T, t) = & \left[ (28.653 + 0.012T^2)A - (32.106 - 0.548T^2)A^{2/3} \right. \\
& \left. - (24.674 - 0.545T^2)I^2A + (171.020 - 2.577T^2)I^2A^{2/3} \right] \\
& + \left[ -(3.878 + 0.007T^2)A + (12.325 + 0.020T^2)A^{2/3} \right. \\
& \left. + (1.332 + 0.031T^2)I^2A - (276.281 + 0.916T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}}
\end{aligned} \tag{11}$$

for SLy4 parameter set,

$$E_K(A, Z, T, t) = \left[ (25.483 + 0.019T^2)A - (26.275 - 0.537T^2)A^{2/3} \right.$$

$$\begin{aligned}
& -(33.636 - 0.548T^2)I^2A + (166.818 - 2.560T^2)I^2A^{2/3} \\
& + \left[ -(0.489 + 0.008T^2)A + (1.409 + 0.021T^2)A^{2/3} \right. \\
& \left. -(1.340 - 0.030T^2)I^2A - (40.353 + 0.940T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}}
\end{aligned} \tag{12}$$

for SkM\* parameter set, and

$$\begin{aligned}
E_K(A, Z, T, t) = & \left[ (19.957 + 0.039T^2)A - (15.845 - 0.500T^2)A^{2/3} \right. \\
& \left. -(6.881 - 0.423T^2)I^2A + (56.508 - 2.061T^2)I^2A^{2/3} \right] \\
& + \left[ -(0.387 + 0.008T^2)A + (1.123 + 0.022T^2)A^{2/3} \right. \\
& \left. -(0.871 - 0.031T^2)I^2A - (31.541 + 0.955T^2)I^2A^{2/3} \right] \frac{t}{A^{1/3}}
\end{aligned} \tag{13}$$

for SkM( $m^* = m$ ) parameter set.

Table I and above Eqs.(8)-(13) show that the contribution of the  $T^2$  dependent term in kinetic energy (see Eq.(A3)) to the neutron skin size dependent coefficients  $E_i(T)_{sk}$  are very small which is consistent with the result in Ref.[13]. In Ref.[13], we estimated approximately the neutron skin size dependence coming from the  $T^2$  term in kinetic energy (see Eq.(A3)) and found the effects were small. The contribution of the  $T^2$  dependent term in kinetic energy to the neutron skin size are insensitive to the Skyrme parameter set used and to the temperature. The neutron skin dependent volume energy coefficient  $E_V(T)_{sk}$  has smallest effect of  $T^2$  dependence and the surface symmetry energy  $S_S(T)_{sk}$  has a largest effect similar to the  $T^2$  dependence of the corresponding skin independent coefficients of energy expansion.

By fitting the values of  $E_i(T)_{sk}$  for temperatures which are  $T = 0, 1, 2$ , and  $3$  MeV in Table I, the  $T$ -dependences of the neutron skin dependent coefficients are

$$\begin{aligned}
E_V(T)_{sk} & [= 5.52016 - 0.00722T^2] \\
& = 5.52016 - 0.00008T - 0.01005T^2 - 0.00007T^3
\end{aligned} \tag{14}$$

$$\begin{aligned}
E_S(T)_{sk} & [= -17.07720 + 0.01959T^2] \\
& = -17.07720 + 0.00091T + 0.03599T^2 + 0.00044T^3
\end{aligned} \tag{15}$$

$$\begin{aligned}
S_V(T)_{sk} & [= 63.20304 + 0.03095T^2] \\
& = 63.20304 + 0.00560T + 0.01927T^2 - 0.00322T^3
\end{aligned} \tag{16}$$

$$\begin{aligned}
S_S(T)_{sk} & [= -711.5588 - 0.91606T^2] \\
& = -711.5588 + 0.00963T - 0.01940T^2 + 0.06117T^3
\end{aligned} \tag{17}$$

for SLy4 parameter set. The first expressions given in the square parenthesis are from the values for  $T = 0$  in Table I same as in Eqs.(8)-(13) for comparison.

$$\begin{aligned}
E_V(T)_{sk} & [= 5.58368 - 0.00761T^2] \\
& = 5.58368 - 0.00042T - 0.01018T^2 - 0.00013T^3
\end{aligned} \tag{18}$$

$$\begin{aligned}
E_S(T)_{sk} & [= -17.26147 + 0.02082T^2] \\
& = -17.26147 + 0.00154T + 0.03763T^2 + 0.00056T^3
\end{aligned} \tag{19}$$

$$\begin{aligned}
S_V(T)_{sk} & [= 62.78658 + 0.02954T^2] \\
& = 62.78658 + 0.00634T + 0.04623T^2 - 0.00358T^3
\end{aligned} \tag{20}$$

$$\begin{aligned}
S_S(T)_{sk} & [= -706.2461 - 0.94030T^2] \\
& = -706.2461 + 0.06100T - 0.15020T^2 + 0.08320T^3
\end{aligned} \tag{21}$$

for SkM\* parameter set.

$$\begin{aligned}
E_V(T)_{sk} & [= 6.98087 - 0.00793T^2] \\
& = 6.98087 - 0.000120T - 0.01048T^2 - 0.00008T^3
\end{aligned} \tag{22}$$

$$\begin{aligned}
E_S(T)_{sk} & [= -21.69093 + 0.022014T^2] \\
& = -21.69093 + 0.00070T + 0.03447T^2 + 0.00026T^3
\end{aligned} \tag{23}$$

$$\begin{aligned}
S_V(T)_{sk} & [= 63.36637 + 0.03144T^2] \\
& = 63.36637 - 0.00421T + 0.02050T^2 - 0.00150667T^3
\end{aligned} \tag{24}$$

$$\begin{aligned}
S_S(T)_{sk} & [= -611.8476 - 0.95456T^2] \\
& = -611.8476 + 0.03905T - 0.54265T^2 + 0.01900T^3
\end{aligned} \tag{25}$$

for SkM( $m^* = m$ ) parameter set.

By fitting the values for  $T = 0, 1, 2$ , and  $3$  MeV in Table I, the temperature  $T$  and neutron skin size  $t$  dependence of the expansion coefficients of the kinetic energy are

$$\begin{aligned}
E_V(T, t)_K & [= (28.65250 + 0.01151T^2) + (-3.87804 - 0.00722T^2)\frac{t}{A^{1/3}}] \\
& = (28.65250 - 0.00801T - 0.04819T^2 - 0.00264T^3) \\
& \quad + (-3.87804 + 0.00117T + 0.01064T^2 + 0.00078T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{26}$$

$$\begin{aligned}
E_S(T, t)_K & [= (-32.10626 + 0.54833T^2) + (12.32541 + 0.01959T^2)\frac{t}{A^{1/3}}] \\
& = (-32.10626 + 0.01492T + 0.45071T^2 + 0.00230T^3) \\
& \quad + (12.32541 - 0.00188T - 0.03149T^2 - 0.00207T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{27}$$

$$\begin{aligned}
S_V(T, t)_K & [= (-24.67405 + 0.54454T^2) + (1.33205 + 0.03095T^2)\frac{t}{A^{1/3}}] \\
& = (-24.67405 - 0.13177T + 0.26686T^2 - 0.02916T^3) \\
& \quad + (1.33205 + 0.03334T - 0.00683T^2 - 0.00072T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{28}$$

$$\begin{aligned}
S_S(T, t)_K & [= (171.0202 - 2.57693T^2) + (-276.2811 - 0.91606T^2)\frac{t}{A^{1/3}}] \\
& = (171.0202 + 1.18275T - 2.36020T^2 + 0.23595T^3) \\
& \quad + (-276.2811 - 0.18403T + 1.15055T^2 + 0.04358T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{29}$$

for SLy4 parameter set.

$$\begin{aligned}
E_V(T, t)_K & [= (25.48323 + 0.01925T^2) + (-0.48934 - 0.00761T^2)\frac{t}{A^{1/3}}] \\
& = (25.48323 + 0.00253T - 0.04169T^2 - 0.00038T^3) \\
& \quad + (-0.48934 + 0.00007T - 0.01019T^2 - 0.00003T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{30}$$

$$\begin{aligned}
E_S(T, t)_K & [= (-26.27463 + 0.53708T^2) + (1.40926 + 0.02082T^2)\frac{t}{A^{1/3}}] \\
& = (-26.27453 - 0.00990T + 0.47594T^2 - 0.00145T^3) \\
& \quad + (1.40926 - 0.00022T + 0.02858T^2 + 0.00008T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{31}$$

$$\begin{aligned}
S_V(T, t)_K & [= (-33.63624 + 0.54816T^2) + (-1.33997 + 0.02954T^2)\frac{t}{A^{1/3}}] \\
& = (-33.63624 - 0.08819T + 0.24295T^2 + 0.00042T^3) \\
& \quad + (-1.33997 - 0.00176T + 0.05211T^2 + 0.00159T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{32}$$

$$\begin{aligned}
S_S(T, t)_K & [= (166.8184 - 2.55977T^2) + (-40.35256 - 0.94030T^2)\frac{t}{A^{1/3}}] \\
& = (166.8184 + 0.07982T - 0.93650T^2 - 0.09752T^3) \\
& \quad + (-40.35256 - 0.00212T - 1.22403T^2 - 0.01232T^3)\frac{t}{A^{1/3}}
\end{aligned} \tag{33}$$

for SkM\* parameter set.

$$E_V(T, t)_K [= (19.95733 + 0.03948T^2) + (-0.38673 - 0.00793T^2)\frac{t}{A^{1/3}}]$$

$$= (19.95733 + 0.00161T + 0.01917T^2 - 0.00003T^3) + (-0.38673 + 0.00006T - 0.00962T^2 - 0.00002T^3) \frac{t}{A^{1/3}} \quad (34)$$

$$\begin{aligned} E_S(T, t)_K & [= (-15.84463 + 0.49952T^2) + (1.12277 + 0.02201T^2) \frac{t}{A^{1/3}}] \\ &= (-15.84463 - 0.00388T + 0.46826T^2 + 0.00162T^3) \\ &\quad + (1.12277 - 0.00024T + 0.02672T^2 + 0.00005T^3) \frac{t}{A^{1/3}} \end{aligned} \quad (35)$$

$$\begin{aligned} S_V(T, t)_K & [= (-6.88128 + 0.42271T^2) + (-0.87085 + 0.03144T^2) \frac{t}{A^{1/3}}] \\ &= (-6.88128 + 0.01595T + 0.34709T^2 - 0.00140T^3) \\ &\quad + (-0.87085 + 0.00273T + 0.04486T^2 + 0.00089T^3) \frac{t}{A^{1/3}} \end{aligned} \quad (36)$$

$$\begin{aligned} S_S(T, t)_K & [= (56.50820 - 2.06058T^2) + (-31.54054 - 0.95456T^2) \frac{t}{A^{1/3}}] \\ &= (56.50820 - 0.15544T - 1.63960T^2 + 0.00843T^3) \\ &\quad + (-31.54054 - 0.02086T - 1.16564T^2 - 0.00796T^3) \frac{t}{A^{1/3}} \end{aligned} \quad (37)$$

for SkM( $m^* = m$ ) parameter set.

Comparing the first line (or Eqs.(8)-(13)) and second line of above Eqs.(14) – (37), we can see the temperature  $T$  dependence of the neutron skin size dependent coefficients  $E_i(T)_{sk}$  comes not only from the  $T^2$  term in the kinetic energy, Eq.(A3), but also from the potential energy through the different saturation density for different temperature. Especially, the neutron skin size dependence of kinetic energy expansion coefficients for SLy4 parameter set have opposite sign in their  $T$  dependence compared with the  $T^2$  term in kinetic energy ( $T^2$  dependence in Eqs.(8)-(13)). Table I and Eqs.(14)-(25) show that the  $T$  dependence of the neutron skin size dependent coefficient is much faster than the  $T^2$  term in kinetic energy of Eq.(A3) alone for volume energy coefficient  $E_V(T)_{sk}$  and surface energy coefficient  $E_S(T)_{sk}$ . By contrast this it is much slower than the  $T^2$  term for volume symmetry energy coefficient  $S_V(T)_{sk}$  and surface symmetry energy coefficient  $S_S(T)_{sk}$  except for volume symmetry energy  $S_V(T)_{sk}$  of the SkM\* parameter set.

Eqs.(26)-(37) show that even the kinetic energy also has an extra  $T$  dependence, beside the  $T^2$  term in kinetic energy of Eq.(A3), through the  $T$  dependence of saturation density. For SLy4 Skyrme interaction, the  $T$  dependence of the kinetic energy part of the neutron skin size dependent energy expansion coefficients  $E_i(T)_{sk}$  has opposite sign with the  $T^2$  term of kinetic energy Eq.(A3). For this interaction the  $T$  dependence of the kinetic energy part of the volume symmetry energy coefficient  $S_V(T)_{sk}$  is rather linear as compared to a  $T^2$  behavior. For SkM\* and SkM( $m^* = m$ ) interactions, the kinetic energy part of the neutron skin dependent energy coefficients  $E_i(T)_{sk}$  has the same sign in its  $T$  dependences with the  $T^2$  term in the kinetic energy of Eq.(A3) but has a faster dependence of  $T$  compared to the  $T^2$  term of Eq.(A3). The neutron skin independent kinetic energy expansion coefficients have a slower  $T$  dependence than a  $T^2$  term for a kinetic energy, Eq.(A3), except the volume energy coefficient  $E_V(T, t = 0)_K$  for SLy4 and SkM\* parameter sets which have an opposite sign compared to  $T^2$  term of kinetic energy Eq.(A3). The neutron skin size independent volume symmetry energy coefficient  $S_V(T, t = 0)_K$  for SLy4 parameter set has a linear  $T$  dependence comparable order to the  $T^2$  dependence.

The kinetic energy expansion at zero  $T$ , from Eqs.(11)-(13) or from Eqs.(26)-(37), are

$$\begin{aligned} E_K(A, Z, T = 0, t) &= \left[ 28.653A - 32.106A^{2/3} - 24.674I^2A + 171.020I^2A^{2/3} \right] \\ &\quad + \left[ -3.878A + 12.325A^{2/3} + 1.332I^2A - 276.281I^2A^{2/3} \right] \frac{t}{A^{1/3}} \end{aligned} \quad (38)$$

for SLy4 parameter set,

$$\begin{aligned} E_K(A, Z, T = 0, t) &= \left[ 25.483A - 26.275A^{2/3} - 33.636I^2A + 166.818I^2A^{2/3} \right] \\ &\quad + \left[ -0.489A + 1.409A^{2/3} - 1.340I^2A - 40.353I^2A^{2/3} \right] \frac{t}{A^{1/3}} \end{aligned} \quad (39)$$

for SkM\* parameter set, and

$$\begin{aligned} E_K(A, Z, T = 0, t) &= \left[ 19.957A - 15.845A^{2/3} - 6.881I^2A + 56.508I^2A^{2/3} \right] \\ &\quad + \left[ -0.387A + 1.123A^{2/3} - 0.871I^2A - 31.541I^2A^{2/3} \right] \frac{t}{A^{1/3}} \end{aligned} \quad (40)$$



for SkM( $m^* = m$ ) parameter set. For an infinite nuclear matter, the surface terms disappear and only the volume terms survive. These results show that the surface symmetry energy coefficient has the largest effect from kinetic energy as compared to the other coefficients. Here we can see the surface kinetic energy and the volume symmetry kinetic energy coefficients are negative. However the total kinetic energy and total symmetry kinetic energy including  $A$  factors are positive. The neutron skin dependent kinetic energies with  $A$  factor included are negative. Compare to result from Fermi gas model,

$$E_K = 12I^2A + 9I^2A^{2/3} \quad (41)$$

which is good for high  $T$  or low density limit without any interaction. Here both the volume and surface symmetry energies are positive. With Skyrme interaction, the isospin dependent part of the effective mass in a finite nuclei may become negative depending on the force parameter and densities of proton and neutron. Thus the signs in Eqs.(38)-(40) result.

#### IV. CONCLUSION AND SUMMARY

Understanding properties of the symmetry energy is important in many area of nuclear physics as mentioned in the introduction. In this paper we studied properties of the symmetry energy, both volume and surface parts, along with other terms which appear in the Weizsacker mass formulae. Our investigation was based on a finite temperature density functional approach. In a finite temperature approach, the dependence of various terms on temperature can be obtained. Energetic probes lead to excited nuclei which may be characterized by a hot liquid drop extension of the Weizsacker mass formulae. We used several different interactions of the Skyrme type to examine the dependence of various quantities on the interaction and associated effective masses that appear. Our analysis in the present study emphasized the role of the neutron skin on various terms that appear in the mass formulae. The radii of protons and neutrons were therefore allowed to be different in a Saxon-Wood form for the density distributions of these particles. We then proceeded to calculate various terms using an expansion about the equal radii point. The corrections that arise from a neutron skin are then proportional to the skin thickness  $t$  over the radius  $R$  or  $t/R \sim t/A^{1/3}$ . The skin thickness  $t/R$  itself can be proportional the neutron excess fraction  $I = (N - Z)/A$  when the proton and neutron central densities are the same. Thus the  $A$  and  $I$  dependences of various terms in the mass formulae are modified.

Table I contains the results for three Skyrme interactions, SLy4, SkM\* and SkM( $m^* = m$ ). Results for the volume energy  $E_V(T)$  and  $E_V(T)_{sk}$ , surface energy  $E_S(T)$  and  $E_S(T)_{sk}$ , volume symmetry energy  $S_V(T)$  and  $S_V(T)_{sk}$ , and surface symmetry energy  $S_S(T)$  and  $S_S(T)_{sk}$  are given. The terms with an additional subscript “sk” are the skin coefficients of Eq.(3). The kinetic energy contributions, labeled “Kine”, to each term are also given. The difference of the total and kinetic term is from the interaction.

The results show that the neutron skin size dependent and independent energy expansion coefficients are rather insensitive to the Skyrme interaction used while the kinetic energy and potential energy expansion coefficients separately are sensitive to the interaction used and somewhat follow the effective mass of the Skyrme parameter. The temperature dependence of the neutron skin size dependent energy expansion coefficients are much more insensitive than the temperature dependence of neutron skin size independent coefficients. The magnitude of the neutron skin size dependent coefficients are largely different for different coefficients compared to the neutron skin size independent coefficients. The neutron skin size dependent volume energy coefficient has the smallest magnitude while the surface symmetry energy coefficient has the largest magnitude. The neutron skin size dependence of surface symmetry energy is much larger than the skin size dependence of the volume symmetry energy with  $A$  dependence included. The surface symmetry kinetic energy coefficients, both the neutron skin size dependent and independent ones, have the largest magnitude of the kinetic energy expansion coefficients. The neutron skin size dependent volume energy coefficient has the smallest temperature dependence and the neutron skin size dependent surface symmetry energy coefficient has the largest temperature dependence similar to the neutron skin independent coefficients. The temperature dependence of the neutron skin size dependent coefficients is smaller than the temperature dependence of the skin size independent coefficients. The temperature dependences of the neutron skin size dependent energy coefficients have a large effect from the different saturation density for different temperature and thus do not follow  $T^2$  behavior of the explicit  $T^2$  dependence of kinetic energy.

Considering the neutron skin size dependence  $t/R$  factor with  $R = r_0A^{1/3}$ , the neutron skin dependent energy expansion coefficients has an extra  $A^{-1/3}$  factor in the energy expansion. On the other hand, if we relate the dimensionless factor  $t/R$  to the isospin factor  $I = (N - Z)/A$  then the energy expansion including the neutron skin size dependence introduce an extra  $I$  factor. With this extra  $I$  dependence we may be able to extract some information on the neutron skin size by expanding the empirical energy of various nuclei with including odd power of  $I$  up to third order.

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### Appendix A: Skyrme interaction

The Hamiltonian for a Skyrme interaction is

$$\begin{aligned}
H(\vec{r}) &= H_B(\vec{r}) + H_S(\vec{r}) + H_C(\vec{r}) \\
H_B &= \frac{\hbar^2}{2m_p}\tau_p + \frac{\hbar^2}{2m_n}\tau_n \\
&\quad + \frac{1}{4} \left[ t_1 \left( 1 + \frac{x_1}{2} \right) + t_2 \left( 1 + \frac{x_2}{2} \right) \right] \rho\tau - \frac{1}{4} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] (\rho_p\tau_p + \rho_n\tau_n) \\
&\quad + \frac{t_0}{2} \left[ \left( 1 + \frac{x_0}{2} \right) \rho^2 - \left( \frac{1}{2} + x_0 \right) (\rho_p^2 + \rho_n^2) \right] \\
&\quad + \frac{t_3}{12} \left[ \left( 1 + \frac{x_3}{2} \right) \rho^2 - \left( \frac{1}{2} + x_3 \right) (\rho_p^2 + \rho_n^2) \right] \rho^\alpha \\
H_S &= \frac{1}{16} \left[ 3t_1 \left( 1 + \frac{x_1}{2} \right) - t_2 \left( 1 + \frac{x_2}{2} \right) \right] (\vec{\nabla}\rho)^2 - \frac{1}{16} \left[ 3t_1 \left( \frac{1}{2} + x_1 \right) + t_2 \left( \frac{1}{2} + x_2 \right) \right] [(\vec{\nabla}\rho_p)^2 + (\vec{\nabla}\rho_n)^2] \\
&= -\frac{1}{16} \left[ 3t_1 \left( 1 + \frac{x_1}{2} \right) - t_2 \left( 1 + \frac{x_2}{2} \right) \right] \rho \nabla^2 \rho + \frac{1}{16} \left[ 3t_1 \left( \frac{1}{2} + x_1 \right) + t_2 \left( \frac{1}{2} + x_2 \right) \right] (\rho_p \nabla^2 \rho_p + \rho_n \nabla^2 \rho_n) \\
H_C &= \frac{e^2}{2} \rho_p(\vec{r}) \int d^3r' \frac{\rho_p(\vec{r}')}{|\vec{r} - \vec{r}'|} - \frac{3e^2}{4} \left( \frac{3}{\pi} \right)^{1/3} \rho_p^{4/3}(\vec{r})
\end{aligned} \tag{A1}$$

The  $H(\vec{r})$  has a bulk part  $H_B(\vec{r})$ , a surface part  $H_S(\vec{r})$  with gradient terms and a Coulomb term  $H_C(\vec{r})$ . The effective mass  $m_q^*$  is

$$\begin{aligned}
\frac{m}{m_q^*} &= 1 + \frac{2m}{\hbar^2} \left\{ \frac{1}{4} \left[ t_1 \left( 1 + \frac{x_1}{2} \right) + t_2 \left( 1 + \frac{x_2}{2} \right) \right] \rho - \frac{1}{4} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] \rho_q \right\} \\
&= 1 + \frac{2m}{\hbar^2} \left\{ \frac{1}{16} [3t_1 + (5 + 4x_2)t_2] \rho \mp \frac{1}{8} \left[ t_1 \left( \frac{1}{2} + x_1 \right) - t_2 \left( \frac{1}{2} + x_2 \right) \right] \rho(2y - 1) \right\}
\end{aligned} \tag{A2}$$

where  $q = n, p$  for neutron or proton. At low  $T$ ,

$$\tau_q(\vec{r}) = \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2 m_q^{*2}}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} T^2 + \dots \right] \tag{A3}$$

The gradient terms are important in finite nuclei and the Coulomb term is important for the charged proton component. The  $t_0, t_1, t_2, t_3$  and  $x_0, x_1, x_2, x_3$  are parameters. Different choices of these parameters give rise to different Skyrme interactions.

### Appendix B: Expansion of density

The central density  $\rho_{qc}$  of the density distribution for a given value of  $R_q$  should be determined to give a fixed number of nucleons  $N_q$ ;

$$N_q(R_q) = \int d^3r \rho_q(\vec{r}) = 4\pi \int_0^\infty r^2 dr \frac{\rho_{qc}(R_q)}{1 + e^{(r-R_q)/a}} = \rho_{qc}(R_q) \frac{4\pi}{3} R_q^3 \left[ 1 + \pi^2 \left( \frac{a}{R_q} \right)^2 \right] \tag{B1}$$

This normalization condition gives up to first order in  $dR_q = (R_q - R)$ ,

$$dN_q = \frac{d\rho_{qc}}{dR} \frac{4\pi}{3} R^3 \left[ 1 + \pi^2 \left( \frac{a}{R} \right)^2 \right] dR_q + \rho_{qc} \frac{4\pi}{3} R^2 \left[ 3 + \pi^2 \left( \frac{a}{R} \right)^2 \right] dR_q = 0 \tag{B2}$$

$$\frac{d\rho_{qc}}{dR} = -\frac{\rho_{qc}(R)}{R} \frac{\left[3 + \pi^2 \left(\frac{a}{R}\right)^2\right]}{\left[1 + \pi^2 \left(\frac{a}{R}\right)^2\right]} \quad (\text{B3})$$

$$\rho_{qc}(R_q) = \rho_{qc}(R) \left[1 - \left(\frac{3 + \pi^2 \left(\frac{a}{R}\right)^2}{1 + \pi^2 \left(\frac{a}{R}\right)^2}\right) \left(\frac{R_q - R}{R}\right) + \dots\right] \quad (\text{B4})$$

Thus the first order correction coefficient to  $\rho_{qc}^m$  due to fixed  $N_q$  is the zeroth order term times  $-\frac{m}{R} \left[\frac{3 + \pi^2 \left(\frac{a}{R}\right)^2}{1 + \pi^2 \left(\frac{a}{R}\right)^2}\right]$ . This result is independent of which type of particle, neutron or proton.

If we keep the particle number  $N_q$  and the total central density  $\rho_c = \rho_{nc} + \rho_{pc}$  to be a constant while varying  $R_q$ , then

$$\begin{aligned} \rho_c(R) &= \rho_{nc}(R_n) + \rho_{pc}(R_p) \\ &= \rho_{nc}(R) + \rho_{pc}(R) - \left[\frac{3 + \pi^2 \left(\frac{a}{R}\right)^2}{1 + \pi^2 \left(\frac{a}{R}\right)^2}\right] \left[\rho_{nc}(R) \left(\frac{R_n - R}{R}\right) + \rho_{pc}(R) \left(\frac{R_p - R}{R}\right)\right] \\ &= \rho_{nc}(R) + \rho_{pc}(R) \end{aligned} \quad (\text{B5})$$

Thus we get

$$\rho_{nc} \left(\frac{R_n - R}{R}\right) + \rho_{pc} \left(\frac{R_p - R}{R}\right) = 0 \quad (\text{B6})$$

and

$$\begin{aligned} R_p - R &= -\frac{\rho_{nc}}{\rho_{pc}}(R_n - R) \\ t = R_n - R_p &= (R_n - R) - (R_p - R) = (R_n - R) \left(1 + \frac{\rho_{nc}}{\rho_{pc}}\right) = \frac{\rho_c}{\rho_{pc}}(R_n - R) \end{aligned} \quad (\text{B7})$$

Finally we have

$$\begin{aligned} t_n &= R_n - R = \frac{\rho_{pc}}{\rho_c} t, \\ t_p &= R_p - R = -\frac{\rho_{nc}}{\rho_c} t \end{aligned} \quad (\text{B8})$$

to lowest order in  $t$ . The same result can be obtained by requiring  $A = N + Z$  constant with keeping the central densities  $\rho_{qc}$  unchanged in the  $dR_q$  expansion. When  $R_q = R$ ,  $\rho_{qc}(R)/\rho_c(R) = N_q/A$  and thus we get

$$\begin{aligned} t_n &= R_n - R = \frac{Z}{A} t, \\ t_p &= R_p - R = -\frac{N}{A} t \end{aligned} \quad (\text{B9})$$

This is the same result given in Ref.[21].

Since the size  $R_q$  depends on the central density  $\rho_{qc}$  for a given value of particle number  $N_q$  as in Eq.(B1), the neutron skin size  $t$  is related to the particle number  $N_q$  and the central density  $\rho_{qc}$ . Using Eqs.(B1) and (B8) we can obtain following conditions to lowest order in  $t$ .

$$\begin{aligned} N - Z &= \frac{4\pi}{3} \left[ (R_n^3 + \pi^2 a^2 R_n) \rho_{nc} - (R_p^3 + \pi^2 a^2 R_p) \rho_{pc} \right] \\ &= \frac{4\pi}{3} \left[ R^3 \left(1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R}\right)^3 \rho_{nc} + \pi^2 a^2 R \left(1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R}\right) \rho_{nc} \right. \\ &\quad \left. - R^3 \left(1 - \frac{\rho_{nc}}{\rho_c} \frac{t}{R}\right)^3 \rho_{pc} - \pi^2 a^2 R \left(1 - \frac{\rho_{nc}}{\rho_c} \frac{t}{R}\right) \rho_{pc} \right] \\ &\approx \frac{4\pi}{3} \left[ R^3 \left(1 + 3 \frac{\rho_{pc}}{\rho_c} \frac{t}{R}\right) \rho_{nc} - R^3 \left(1 - 3 \frac{\rho_{nc}}{\rho_c} \frac{t}{R}\right) \rho_{pc} \right] \end{aligned}$$

$$\begin{aligned}
& +\pi^2 a^2 R \left(1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R}\right) \rho_{nc} - \pi^2 a^2 R \left(1 - \frac{\rho_{nc}}{\rho_c} \frac{t}{R}\right) \rho_{pc} \Big] \\
& = \frac{4\pi}{3} (R^3 + \pi^2 a^2 R) (\rho_{nc} - \rho_{pc}) + \frac{4\pi}{3} (3R^3 + \pi^2 a^2 R) \left( \frac{\rho_{nc}}{\rho_c} \rho_{pc} + \frac{\rho_{pc}}{\rho_c} \rho_{nc} \right) \frac{t}{R} \\
& = \frac{4\pi}{3} R^3 \left[ 1 + \pi^2 \left( \frac{a}{R} \right)^2 \right] \rho_c \left( \frac{\rho_{nc} - \rho_{pc}}{\rho_c} \right) + \frac{4\pi}{3} \left[ R^3 + \frac{\pi^2 a^2}{3} R \right] 3 \left( \frac{\rho_{nc}}{\rho_c} \rho_{pc} + \frac{\rho_{pc}}{\rho_c} \rho_{nc} \right) \frac{t}{R} \\
& = A \left( \frac{\rho_{nc} - \rho_{pc}}{\rho_c} \right) + 3 \frac{A}{\rho_c} \left( \frac{\rho_{nc}}{\rho_c} \rho_{pc} + \frac{\rho_{pc}}{\rho_c} \rho_{nc} \right) \left[ \frac{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2}{1 + \left( \frac{\pi a}{R} \right)^2} \right] \frac{t}{R} \tag{B10}
\end{aligned}$$

$$\begin{aligned}
\frac{t}{R} & = \frac{1}{3} \frac{1}{A} \frac{\rho_c}{\left( \frac{\rho_{nc}}{\rho_c} \rho_{pc} + \frac{\rho_{pc}}{\rho_c} \rho_{nc} \right)} \left[ (N - Z) - A \left( \frac{\rho_{nc} - \rho_{pc}}{\rho_c} \right) \right] \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] \\
& = \frac{1}{3} \left( \frac{1}{2(1 - y_c)y_c} \right) \left[ \left( \frac{N - Z}{A} \right) - (1 - 2y_c) \right] \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] \tag{B11}
\end{aligned}$$

where  $y_c = \rho_{pc}/\rho_c$  is the proton fraction of the central density. For the case of  $\rho_{nc} \approx \rho_{pc}$ ,  $y_c = 1/2 + \epsilon$  and  $(1 - y_c) = 1/2 - \epsilon$ . Then the factor  $[2(1 - y_c)y_c]^{-1}$  becomes

$$\frac{1}{2(1 - y_c)y_c} = \frac{2}{(1 - 2\epsilon)(1 + 2\epsilon)} \approx 2(1 - 4\epsilon^2) \tag{B12}$$

Thus for an uniform distribution (diffuseness parameter  $a = 0$ ) with  $\rho_{nc} \approx \rho_{pc}$ , the neutron skin size  $t/R$  of Eq.(B11) becomes, up to first order in  $\epsilon$ ,

$$\frac{t}{R} = \frac{2}{3} \frac{N - Z}{A} - \frac{2}{3} (1 - 2y_c) \tag{B13}$$

which is the result given in Ref.[21]. On the other hand, as another form,

$$\begin{aligned}
\frac{N}{\rho_{nc}} - \frac{Z}{\rho_{pc}} & = \frac{4\pi}{3} [(R_n^3 + \pi^2 a^2 R_n) - (R_p^3 + \pi^2 a^2 R_p)] \\
& = \frac{4\pi}{3} \left[ R^3 \left( 1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R} \right)^3 - R^3 \left( 1 - \frac{\rho_{nc}}{\rho_c} \frac{t}{R} \right)^3 + \pi^2 a^2 R \left( 1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R} \right) - \pi^2 a^2 R \left( 1 - \frac{\rho_{nc}}{\rho_c} \frac{t}{R} \right) \right] \\
& \approx \frac{4\pi}{3} \left[ R^3 \left( 1 + 3 \frac{\rho_{pc}}{\rho_c} \frac{t}{R} \right) - R^3 \left( 1 - 3 \frac{\rho_{nc}}{\rho_c} \frac{t}{R} \right) + \pi^2 a^2 R \left( 1 + \frac{\rho_{pc}}{\rho_c} \frac{t}{R} \right) - \pi^2 a^2 R \left( 1 - \frac{\rho_{nc}}{\rho_c} \frac{t}{R} \right) \right] \\
& = \frac{4\pi}{3} \left[ R^3 3 \left( \frac{\rho_{pc}}{\rho_c} + \frac{\rho_{nc}}{\rho_c} \right) + \pi^2 a^2 R \left( \frac{\rho_{pc}}{\rho_c} + \frac{\rho_{nc}}{\rho_c} \right) \right] \frac{t}{R} = \frac{4\pi}{3} 3 \left[ R^3 + \frac{\pi^2 a^2}{3} R \right] \frac{t}{R} \\
& = 3 \frac{A}{\rho_c} \left[ \frac{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2}{1 + \left( \frac{\pi a}{R} \right)^2} \right] \frac{t}{R} \tag{B14}
\end{aligned}$$

$$\frac{t}{R} = \frac{1}{3} \frac{\rho_c}{A} \left( \frac{N}{\rho_{nc}} - \frac{Z}{\rho_{pc}} \right) \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] = \frac{1}{3} \frac{1}{A} \left( \frac{N}{1 - y_c} - \frac{Z}{y_c} \right) \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] \tag{B15}$$

Even if Eqs.(B11) and (B15) look different they are the same equation. The proton ratio  $y_c$  has the range of  $Z/A \leq y_c \leq 1/2$  for finite nuclei where  $y_c = Z/A$  when  $R_n = R_p$  and  $y_c = 1/2$  for  $\rho_{nc} = \rho_{pc}$ .

For one extreme case of the same central density  $\rho_{nc} = \rho_{pc}$ , the proton ratio  $y = 1/2$  and

$$\frac{t}{R} = \frac{2}{3} \left( \frac{N - Z}{A} \right) \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] = \frac{2}{3} \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] I \tag{B16}$$

Thus the neutron skin size is linearly proportional to the isospin factor  $I$  when the proton and neutron central densities are same. For the other extreme case of the same size  $R_n = R_p = R$ , the proton ratio is  $y_c = Z/A$  with  $1 - y_c = N/A$ , and thus the neutron skin size becomes  $t/R = 0$ . However when  $R_n \approx R_p$  with  $y_c = Z/A + \epsilon$ , we have  $1 - y_c = N/A - \epsilon$

and, from Eq.(B11),

$$\begin{aligned}
\frac{t}{R} &= \frac{1}{3} \left[ \frac{1}{2(N/A - \epsilon)(Z/A + \epsilon)} \right] \left[ \left( \frac{N-Z}{A} \right) - \left( \frac{N}{A} - \epsilon - \frac{Z}{A} - \epsilon \right) \right] \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] \\
&\approx \frac{1}{3} \left[ \frac{1}{2(N/A)(Z/A)} \right] \left( 1 + \frac{A}{N} \epsilon \right) \left( 1 - \frac{A}{Z} \epsilon \right) \left[ \left( \frac{N-Z}{A} \right) - \left( \frac{N-Z}{A} \right) + 2\epsilon \right] \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] \\
&\approx \frac{1}{3} \left( \frac{A^2}{NZ} \right) \left[ \frac{1 + \left( \frac{\pi a}{R} \right)^2}{1 + \frac{1}{3} \left( \frac{\pi a}{R} \right)^2} \right] \epsilon
\end{aligned} \tag{B17}$$

up to the first order in  $\epsilon$ .

Using Eq.(B4), the Fermi density Eq.(1) is expanded up to first order in  $(R_q - R)$  as

$$\begin{aligned}
\rho_q(r) &= \frac{\rho_{qc}(R_q)}{1 + e^{(r-R_q)/a}} = \frac{\rho_{qc}(R_q)}{1 + e^{(y+(R-R_q)/a)}} \\
&= \frac{\rho_{qc}(R)}{1 + e^y} \left[ 1 - \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) + \dots \right] \left[ 1 - \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \dots \right] \\
&= \left( \frac{\rho_{qc}(R)}{1 + e^y} \right) \left[ 1 - \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) - \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \dots \right] \\
&= \left( \frac{\rho_{qc}(R)}{1 + e^y} \right) \left[ 1 + \left( \frac{e^y}{1 + e^y} \frac{R}{a} - \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \dots \right]
\end{aligned} \tag{B18}$$

$$\rho_q^m(r) = \left( \frac{\rho_{qc}(R)}{1 + e^y} \right)^m \left[ 1 + m \left( \frac{e^y}{1 + e^y} \frac{R}{a} - \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \dots \right] \tag{B19}$$

where  $y = (r - R)/a$ . Since  $t_n = \frac{Z}{A}t$  and  $t_p = -\frac{N}{Z}t$  (Eq.(B9)) with  $\rho_{qc}(R)/\rho_c(R) = N_q/A$ ,

$$\begin{aligned}
\rho_n^m(r) + \rho_p^m(R) &= \left( \frac{\rho_c(R)}{1 + e^{(r-R)/a}} \right)^m \left[ \left( \frac{\rho_{nc}^m(R) + \rho_{pc}^m(R)}{\rho_c^m(R)} \right) \right. \\
&\quad \left. + m \left( \frac{e^y}{1 + e^y} \frac{R}{a} - \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{\rho_{nc}^m(R)Z - \rho_{pc}^m(R)N}{\rho_c^m(R)A} \right) \frac{(R_n - R_p)}{R} + \dots \right] \\
&\approx \left( \frac{\rho_c(R)}{1 + e^y} \right)^m \left[ \left( \frac{N^m + Z^m}{A^m} \right) + m \left( \frac{e^y}{1 + e^y} \frac{R}{a} - \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{N^m Z - Z^m N}{A^{m+1}} \right) \frac{t}{R} + \dots \right]
\end{aligned} \tag{B20}$$

From this result it is easy to show that the quantity  $\rho_n^m(r) + \rho_p^m(r)$  has a first order correction from skin size  $t$  and the first order correction vanishes for  $m = 1$ . That is the total density  $\rho(r) = \rho_n(r) + \rho_p(r)$  is independent of the neutron skin size  $t = R_n - R_p$  up to first order. Thus only the explicit  $\rho_q$  dependent terms in Skyrme interaction, not the total density  $\rho$  dependent terms, depend on the skin size  $t$  up to the first order.

### Appendix C: Integral of density functional

Since  $F[\rho_q] = \int d^3r f(\rho_q(\vec{r}))$  for a Fermi density  $\rho_q(r) = \frac{\rho_{qc}}{1 + e^{(r-R_q)/a}}$ , where  $f(\rho_q)$  is a function of a single density  $\rho_q$  only, can be integrated exactly as a function  $F(R_q)$  of  $R_q$  [13], we can expand  $F(R_q)$  easily in terms of  $t_q = R_q - R$  or  $x_q = -t_q/a$  around  $F(R)$ . That is

$$F(R_q) = F(R) + \frac{dF(R)}{dR} t_q + \frac{d^2 F(R)}{dR^2} \frac{t_q^2}{2} + \dots \tag{C1}$$

This is much simpler than using previous method of expanding in  $t$  (Eq.(B18)) first then integrate the results. For fixed  $N_q$ , the central density  $\rho_{qc}^m(R_q)$  is expanded as

$$\rho_{qc}^m(R_q) = \rho_{qc}^m(R) \left[ 1 - m \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \dots \right] \tag{C2}$$

Thus

$$\begin{aligned}
\int d^3r \rho_q^2(\vec{r}) &= \frac{4\pi}{3} \rho_{qc}^2(R_q) (R_q^3 - 3aR_q^2 + \pi^2 a^2 R_q - \pi^2 a^3) \\
&= \frac{4\pi}{3} R_q^3 \rho_{qc}^2(R) \left[ 1 - 3 \left( \frac{a}{R} \right) + \pi^2 \left( \frac{a}{R} \right)^2 - \pi^2 \left( \frac{a}{R} \right)^3 \right] \left[ 1 - 2 \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R_q^3 \rho_{qc}^2(R) \frac{a}{R} \left[ 3 - 6 \left( \frac{a}{R} \right) + \pi^2 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) + \dots
\end{aligned} \tag{C3}$$

$$\begin{aligned}
\int d^3r \rho_q(\vec{r}) \nabla^2 \rho_q(\vec{r}) &= -\frac{4\pi}{3} R_q^3 \frac{\rho_{qc}^2(R_q)}{2R_q a} \left[ 1 + \left( \frac{\pi^2}{3} - 2 \right) \left( \frac{a}{R_q} \right)^2 \right] \\
&= -\frac{4\pi}{3} R_q^3 \frac{\rho_{qc}^2(R)}{2Ra} \left[ 1 + \left( \frac{\pi^2}{3} - 2 \right) \left( \frac{a}{R} \right)^2 \right] \left[ 1 - 2 \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad - \frac{4\pi}{3} R_q^3 \frac{\rho_{qc}^2(R)}{R^2} \left( \frac{R_q - R}{a} \right) + \dots
\end{aligned} \tag{C4}$$

$$\begin{aligned}
\int d^3r \rho_q^{4/3}(\vec{r}) &= \frac{4\pi}{3} R_q^3 \rho_{qc}^{4/3}(R_q) \left[ 1 - 1.335546875 \left( \frac{a}{R_q} \right) + 8.81615625 \left( \frac{a}{R_q} \right)^2 - 5.0303125 \left( \frac{a}{R_q} \right)^3 \right] \\
&= \frac{4\pi}{3} R_q^3 \rho_{qc}^{4/3}(R) \left[ 1 - 1.335546875 \left( \frac{a}{R} \right) + 8.81615625 \left( \frac{a}{R} \right)^2 - 5.0303125 \left( \frac{a}{R} \right)^3 \right] \\
&\quad \times \left[ 1 - \frac{4}{3} \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R_q^3 \rho_{qc}^{4/3}(R) \frac{a}{R} \left[ 3 - 2.67109375 \left( \frac{a}{R} \right) + 8.81615625 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) + \dots
\end{aligned} \tag{C5}$$

$$\begin{aligned}
\int d^3r \rho_p^{5/3}(\vec{r}) &= \frac{4\pi}{3} R_q^3 \rho_{qc}^{5/3}(R_q) \left[ 1 - 2.276943 \left( \frac{a}{R_q} \right) + 9.10458 \left( \frac{a}{R_q} \right)^2 - 7.80506 \left( \frac{a}{R_q} \right)^3 \right] \\
&= \frac{4\pi}{3} R_q^3 \rho_{qc}^{5/3}(R) \left[ 1 - 2.276943 \left( \frac{a}{R} \right) + 9.10458 \left( \frac{a}{R} \right)^2 - 7.80506 \left( \frac{a}{R} \right)^3 \right] \\
&\quad \times \left[ 1 - \frac{5}{3} \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R_q^3 \rho_{qc}^{5/3}(R) \frac{a}{R} \left[ 3 - 4.553886 \left( \frac{a}{R} \right) + 9.10458 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) + \dots
\end{aligned} \tag{C6}$$

$$\begin{aligned}
\int d^3r \rho_p^{8/3}(\vec{r}) &= \frac{4\pi}{3} R_q^3 \rho_{qc}^{8/3}(R_q) \left[ 1 - 4.07693333 \left( \frac{a}{R_q} \right) + 11.836907 \left( \frac{a}{R_q} \right)^2 - 13.26781 \left( \frac{a}{R_q} \right)^3 \right] \\
&= \frac{4\pi}{3} R_q^3 \rho_{qc}^{8/3}(R) \left[ 1 - 4.07693333 \left( \frac{a}{R} \right) + 11.836907 \left( \frac{a}{R} \right)^2 - 13.26781 \left( \frac{a}{R} \right)^3 \right] \\
&\quad \times \left[ 1 - \frac{8}{3} \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R_q^3 \rho_{qc}^{8/3}(R) \frac{a}{R} \left[ 3 - 8.15386666 \left( \frac{a}{R} \right) + 11.836907 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) + \dots
\end{aligned} \tag{C7}$$

$$\begin{aligned}
E_C &= \frac{3}{5} \frac{Z^2 e^2}{R_p} \left[ 1 - \left( \frac{7\pi^2}{6} \right) \left( \frac{a}{R_p} \right)^2 \right] \\
&= \frac{3}{5} \frac{Z^2 e^2}{R} \left[ 1 - \left( \frac{7\pi^2}{6} \right) \left( \frac{a}{R} \right)^2 \right] - \frac{3}{5} \frac{Z^2 e^2}{R} \frac{a}{R} \left[ 1 - \left( \frac{7\pi^2}{2} \right) \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_p - R}{a} \right) + \dots
\end{aligned} \tag{C8}$$

In Ref.[13], the Coulomb exchange term is shown only upto 0th order in  $a/R$  but the actual calculation included all terms of  $a/R$  up to 3. For the term with mixed densities, we need to integrate after expansion. Up to 1st order in

$$x = -t/a,$$

$$\begin{aligned}
\int d^3r \rho_q^m(\vec{r}) \rho^\alpha(\vec{r}) &= 4\pi \rho_{qc}^m(R_q) \rho_c^\alpha(R) \int_0^\infty r^2 dr \left( \frac{1}{1 + e^{(r-R_q)/a}} \right)^m \left( \frac{\rho_{nc}(R_n)/\rho_c(R)}{1 + e^{(r-R_n)/a}} + \frac{\rho_{pc}(R_p)/\rho_c(R)}{1 + e^{(r-R_p)/a}} \right)^\alpha \\
&\approx 4\pi \rho_{qc}^m(R) \rho_c^\alpha(R) \int_0^\infty r^2 dr \left( \frac{1}{1 + e^{(r-R)/a}} \right)^{\alpha+m} \left[ 1 - m \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad - 4\pi \rho_{qc}^m(R) \rho_c^\alpha(R) \int_0^\infty r^2 dr \left( \frac{1}{1 + e^{(r-R)/a}} \right)^{\alpha+m} \left( \frac{e^{(r-R)/a}}{1 + e^{(e-R)/a}} \right) m \left( \frac{R - R_q}{a} \right) \\
&= \frac{4\pi}{3} \rho_{qc}^m(R) \rho_c^\alpha(R) \int_{-\infty}^\infty dy (ay + R)^3 \frac{(\alpha + m)e^y}{(1 + e^y)^{\alpha+m+1}} \left[ 1 - m \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + 4m\pi \rho_{qc}^m(R) \rho_c^\alpha(R) a \int_{-\infty}^\infty dy (ay + R)^2 \frac{e^y}{(1 + e^y)^{\alpha+m+1}} \left( \frac{R_q - R}{a} \right) \quad (C9)
\end{aligned}$$

For SLy4 parameter with  $\alpha = 1/6$  and  $m = 2$ ,

$$\begin{aligned}
\int d^3r \rho_q^2(\vec{r}) \rho^{1/6}(\vec{r}) &= \frac{4\pi}{3} R^3 \rho_{qc}^2 \rho_c^{1/6} \left[ 1 - 3.30669 \left( \frac{a}{R} \right) + 10.331 \left( \frac{a}{R} \right)^2 - 10.7804 \left( \frac{a}{R} \right)^3 \right] \\
&\quad \times \left[ 1 - 2 \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R^3 \rho_{qc}^2 \rho_c^{1/6} \frac{36}{13} \frac{a}{R} \left[ 1 - 2.204466 \left( \frac{a}{R} \right) + 3.4436602 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) \quad (C10)
\end{aligned}$$

Here  $36/13 = 3 \times 2/(13/6)$ . For SkM( $m^* = m$ ) parameter with  $\alpha = 1$  and  $m = 2$ ,

$$\begin{aligned}
\int d^3r \rho_q^2(\vec{r}) \rho(\vec{r}) &= \frac{4\pi}{3} R^3 \rho_{qc}^2 \rho_c \left[ 1 - \frac{9}{2} \left( \frac{a}{R} \right) + (3 + \pi^2) \left( \frac{a}{R} \right)^2 - \frac{3\pi^2}{2} \left( \frac{a}{R} \right)^3 \right] \\
&\quad \times \left[ 1 - 2 \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R^3 \rho_{qc}^2 \rho_c 2 \frac{a}{R} \left[ 1 - 3 \left( \frac{a}{R} \right) + \left( 1 + \frac{\pi^2}{3} \right) \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) \quad (C11)
\end{aligned}$$

Here  $2 = 3 \times 2/3$ . For  $T$ -independent term in kinetic energy with  $\alpha = 1$  and  $m = 5/3$ ,

$$\begin{aligned}
\int d^3r \rho_q^{5/3}(\vec{r}) \rho(\vec{r}) &= \frac{4\pi}{3} R^3 \rho_{qc}^{5/3} \rho_c \left[ 1 - 4.07693333 \left( \frac{a}{R} \right) + 11.836907 \left( \frac{a}{R} \right)^2 - 13.26781 \left( \frac{a}{R} \right)^3 \right] \\
&\quad \times \left[ 1 - \frac{5}{3} \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] \\
&\quad + \frac{4\pi}{3} R^3 \rho_{qc}^{5/3} \rho_c \frac{15}{8} \frac{a}{R} \left[ 1 - 2.71797333 \left( \frac{a}{R} \right) + 3.9456266667 \left( \frac{a}{R} \right)^2 \right] \left( \frac{R_q - R}{a} \right) \quad (C12)
\end{aligned}$$

Here  $(15/8) = 3 \times (5/3)/(8/3)$ . For kinetic energy we cannot use this method since it uses numerical integration.

The kinetic energy has term with the form of  $\rho_q^m m_q^{*2}$  where the effective mass has the form of  $m_q/m_q^* = 1 + a\rho + b\rho_q$ . Since

$$\rho_q^m(r) = \left( \frac{\rho_{qc}(R)}{1 + e^y} \right)^m \left[ 1 - m \left( \frac{e^y}{1 + e^y} \right) \left( \frac{R - R_q}{a} \right) - m \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) + \dots \right] \quad (C13)$$

and  $\rho$  is  $t$  independent up to the first order in the neutron skin  $t$ ,

$$\begin{aligned}
\rho_q^m m_q^{*n} &= \rho_q^m m_q^n (1 + a\rho + b\rho_q)^{-n} \\
&= \left( \frac{\rho_{qc}(R_q)}{1 + e^y} \right)^m m_q^n \left[ 1 - m \left( \frac{e^y}{1 + e^y} \right) x_q + \dots \right] \left[ 1 + a\rho + b \left( \frac{\rho_{qc}(R_q)}{1 + e^y} \right) \left( 1 - \left( \frac{e^y}{1 + e^y} \right) x_q + \dots \right) \right]^{-n}
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{\rho_{qc}(R)}{1+e^y} \right)^m m_q^n \left[ 1 + a\rho + b \left( \frac{\rho_{qc}(R)}{1+e^y} \right) \right]^{-n} \\
&\quad \times \left\{ 1 - \left[ m - \frac{nb \frac{\rho_{qc}(R)}{1+e^y}}{\left( 1 + a\rho + b \frac{\rho_{qc}(R)}{1+e^y} \right)} \right] \left[ \left( \frac{e^y}{1+e^y} \right) x_q + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] + \dots \right\} \\
&= \rho_q^m m_q^{*n} \Big|_{x_q=0} \left\{ 1 - \left[ m - nb\rho_q \frac{m_q^*}{m_q} \Big|_{x_q=0} \right] \left[ \left( \frac{e^y}{1+e^y} \right) x_q + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] + \dots \right\} \quad (C14)
\end{aligned}$$

Since the  $m_q^*$  factor depends on  $\rho_q$ , the first order term in  $x = -t/a$  of  $(\rho_n^m m_n^{*n} + \rho_p^m m_p^{*n})$  does not become zero for any  $m$  for non-zero  $n$ . Similarly total kinetic energy  $\tau = \tau_n + \tau_p$  has nonzero first order term in the neutron skin  $t$  in contrast to the total density  $\rho = \rho_n + \rho_p$  which is independent to  $t$  up to first order. Total kinetic energy is

$$\begin{aligned}
\tau &= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \left( \rho_n^{5/3} + \rho_p^{5/3} \right) + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \left( \rho_n^{1/3} m_n^{*2} + \rho_p^{1/3} m_p^{*2} \right) + \dots \right] \\
&= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \left( \rho_n^{5/3} + \rho_p^{5/3} \right) + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \left( \rho_n^{1/3} m_n^{*2} + \rho_p^{1/3} m_p^{*2} \right) \right]_{x=0} \\
&\quad - \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5}{3} \rho_n^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_n^{1/3} m_n^{*2} \left( \frac{1}{3} - 2b\rho_n \frac{m_n^*}{m_n} \right) T^2 \right]_{x=0} \\
&\quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R - R_n}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_n - R}{R} \right) \right] \\
&\quad - \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5}{3} \rho_p^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_p^{1/3} m_p^{*2} \left( \frac{1}{3} - 2b\rho_p \frac{m_p^*}{m_p} \right) T^2 \right]_{x=0} \\
&\quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R - R_p}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_p - R}{R} \right) \right] + \dots \quad (C15)
\end{aligned}$$

with

$$\begin{aligned}
\tau_q &= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^{*2} + \dots \right] \\
&= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^{*2} \right]_{t=0} \\
&\quad - \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5}{3} \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^{*2} \left( \frac{1}{3} - 2b\rho_q \frac{m_q^*}{m_q} \right) T^2 \right]_{t=0} \\
&\quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R - R_q}{a} \right) + \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] + \dots \\
&= \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^{*2} \right]_{t=0} \\
&\quad + \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5}{3} \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^{*2} \left( \frac{1}{3} - 2b\rho_q \frac{m_q^*}{m_q} \right) T^2 \right]_{t=0} \\
&\quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R_q - R}{a} \right) - \left( \frac{3 + \pi^2 \left( \frac{a}{R} \right)^2}{1 + \pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q - R}{R} \right) \right] + \dots \quad (C16)
\end{aligned}$$

and

$$\begin{aligned}
\rho_q \tau_q &= \rho_q \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^{*2} + \dots \right] \\
&= \rho_q \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^{*2} \right]_{x=0}
\end{aligned}$$



$$\begin{aligned}
& -\rho_q \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{8}{3} \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^{*2} \left( \frac{4}{3} - 2b\rho_q \frac{m_q^*}{m_q} \right) T^2 \right]_{x=0} \\
& \quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R-R_q}{a} \right) + \left( \frac{3+\pi^2 \left( \frac{a}{R} \right)^2}{1+\pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q-R}{R} \right) \right] + \dots \\
& = \rho_q \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} T^2 \rho_q^{1/3} m_q^{*2} \right]_{t=0} \\
& \quad + \rho_q \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \frac{5}{3} \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^{*2} \left( \frac{1}{3} - 2b\rho_q \frac{m_q^*}{m_q} \right) T^2 \right]_{t=0} \\
& \quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R_q-R}{a} \right) - \left( \frac{3+\pi^2 \left( \frac{a}{R} \right)^2}{1+\pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q-R}{R} \right) \right] \\
& \quad + \rho_q \frac{3}{5} \left( \frac{6\pi^2}{\gamma} \right)^{2/3} \left[ \rho_q^{5/3} + \frac{5\pi^2}{3\hbar^4} \left( \frac{\gamma}{6\pi^2} \right)^{4/3} \rho_q^{1/3} m_q^{*2} T^2 \right]_{t=0} \\
& \quad \times \left[ \left( \frac{e^y}{1+e^y} \right) \left( \frac{R_q-R}{a} \right) - \left( \frac{3+\pi^2 \left( \frac{a}{R} \right)^2}{1+\pi^2 \left( \frac{a}{R} \right)^2} \right) \left( \frac{R_q-R}{R} \right) \right] + \dots \tag{C17}
\end{aligned}$$

In Weizacker mass formular it might be better expanding in terms of  $(R_n - R_p)/R$  rather than  $(R_n - R_p)/a$  since the first one is independent of  $A$  while the second one is dependent on  $A$ .

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